(1/2)

Lecture 7, Symplectic Resolutions and Singularities.

I. More on quiver varieties: Notation:

$\boxed{m}$ means dimension m at a framing vertex

ⓝ means dim n at an original vertex. "gauge"

$$\boxed{n} - ①$$ means $T^*\mathbb{C}^n /\!/\!/ \mathbb{C}^\times = GL_1$ $\rightsquigarrow$ get $\{(v,\varphi) \mid \varphi(v)=0\}/\!/\mathbb{C}^\times$

$$\overline{\text{min nil orbit}} \cong \{A \in \mathrm{Mat}_n \mid A^2=0, \mathrm{rk}\, A = 1\},$$

under map $(v,\varphi) \longmapsto \varphi\otimes v \in V^*\otimes V \cong \mathrm{Mat}_n$

$V := \mathbb{C}^n$. ~~$n = \dim V$.~~

Then $T^*\mathbb{C}^n /\!/\!/_\theta\, \mathbb{C}^\times \cong T^*\mathbb{P}^{n-1} \twoheadrightarrow \overline{\min(sl_n)} \cong T^*\mathbb{C}^n /\!/\!/ \mathbb{C}^\times$

parabolic Springer resolution.

Also, $\mathbb{C}^n /\!/_\theta\, \mathbb{C}^\times \cong \mathbb{P}^{n-1}$!

$$\boxed{n} - (n-1) - (n-2) - (n-3) \cdots - ①$$: similarly get flags:

Ordinary GIT qt: $\{\mathbb{C}^n \supsetneq V_{n-1} \supset V_{n-2} \supset \cdots \supset V_1 \supset 0\}/\cong$ $\cong G/B$, $\dim V_i = i$, $\forall i$.

$G = SL_n$, B = Borel.

Quiver variety (symplectic version): $T^*(G/B) \cong \mathfrak{M}_{0,\theta}$ for suitable θ.

$$\mathfrak{M}_{0,\theta} \cong T^*G/B \xrightarrow{\text{Springer}} \mathrm{Nil}(\mathfrak{g}) \cong \mathfrak{M}_{0,0}$$

⚠ Can't do this for other types: it's a "coincidence" that we use a type A quiver to get nilpotent orbit closures also in type A.

Nilpotent orbit closures in so_n or sp_{2n}: Obtained by Hamiltonian reduction by orthogonal + symplectic groups.

E.g.: "$\boxed{SO(2n)} - (Sp(2))$" or "$\boxed{D_n} - (C_1)$": get $(\mathbb{C}^{2n}\otimes\mathbb{C}^2)/\!/\!/ Sp_2 \cong$ $\boxed{\text{min nilpotent orbit closure in } so(2n)}$.

Gen: $\begin{array}{c} \boxed{n} \\ | \\ \boxed{m_1} - \boxed{m_2} - \cdots - \boxed{m_k} \end{array}$ $n > m_1 > m_2 > \dots > m_k \rightsquigarrow T^*(\text{flags } \mathbb{C}^n \supset V_{m_1} \supset \dots \supset V_{m_k})$.

Also: given two orbit closures, labeled by two partitions of n, ($n > m_1 > m_2 \dots > m_k \longleftrightarrow (n-m_1, m_1-m_2, \dots, m_{k-1}-m_k, m_k)$ partition of n up to ordering)

$\overline{\mathcal{O}_\mu} \supseteq \overline{\mathcal{O}_\nu}$

can get the transverse "Slodowy" slice:

$$\boxed{S_{\mu,\nu} := \overline{\mathcal{O}_\mu} \cap \mathrm{Slice}(\mathcal{O}_\nu)}$$

also called "S3" variety after Slodowy, Spaltenstein, and Springer.

Again, $S_{\mu,\nu}$ is a type A quiver variety and has a resolution by varying θ. $\rightsquigarrow$ given by "quiver subtraction" quiver$(\overline{\mathcal{O}_\mu})$ − quiver$(\overline{\mathcal{O}_\nu})$ + framing

II. Symplectic Duality: Idea:

(special) symplectic ~~leaves~~ closures $\longleftrightarrow$ transverse slices to "complementary" symplectic leaves

$$\begin{array}{ccc} X & \longleftrightarrow & X^! \\ \cup & & \cup \\ \text{leaf closure} & \longleftrightarrow & \text{transverse slice} \end{array}$$

It is from 3D mirror symmetry, closely related to string theory + supersymmetry.

Hyperplane arrangements:

Thm (Namikawa) If X is a conical symplectic singularity, then $HP^2(X) \cong H^2(\tilde{X})/W$ parameterises the universal filtered Poisson deformation, $\tilde{X} \to X$ is a S.R. (or $\mathbb{Q}$-factorial terminalisation, aka "minimal model") $\to$ in particular when allowing singularities, $\tilde{X}$ is still smooth outside codim ≥ 4).

Kaledin: Explains for $\tilde{X}$ S.R. how to construct a "twistor deformation" from a line bundle L on $\tilde{X}$ (corr. to $c_1(L) \in H^2(\tilde{X})$)

↳ gives 1-parameter deformation $\tilde{\mathcal{X}}, \mathcal{L}$ of $\tilde{X}, L$, symplectic form on total space of $\mathcal{L}$

Rmk: "Twistor" comes from hyperkähler geometry: if $c_1(L) = [\text{Kähler form of HK metric}]$ $\to$ can extend to "twistor space" over $\mathbb{P}^1$.

Here, W = a product of Weyl groups, one for each codim 2 singularity of X (type ADE ~~on~~ singularity $\Rightarrow$ type A/B, D/C, E/F/G Weyl group)

$H^2(\tilde{X}) = \bigoplus$ of reflection representations
parameterises deformations of $\tilde{X}$

$HP^2(X) \cong H^2(\tilde{X})/W$ (Rmk: reflection rep / Weyl group $\cong$ vector space of some dim
parameterizes deformations e.g.: $\mathbb{C}^n/S_n \cong \mathbb{C}^n \Rightarrow \mathbb{C}^{n-1}/S_n \cong \mathbb{C}^{n-1}$)
of X. Get diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ H^2(\tilde{X}) \arrow[r, two heads] & H^2(\tilde{X})/W \end{tikzcd}$$

Base of universal ~~deformation~~ filtered deformation.

Ex: Grothendieck–Springer:

$$\begin{tikzcd} G \times_B \mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{h} \arrow[r, two heads] & \mathfrak{h}/W \end{tikzcd}$$

$\mathfrak{h} = H^2(T^*G/B) = H^2(G/B)$

Namikawa hyperplane arrangement:

$\mathcal{D} := \{ b \in H^2(\tilde{X}) \mid X_b \text{ is singular} \}$

E.g. $\cdot$ $X = Nil(sl_2) \rightsquigarrow X_b = \{ A \in sl_2 \mid \det A = b^2 \}$
$\Rightarrow \mathcal{D} = \{0\} \subseteq H^2(T^*\mathbb{P}^1) \cong \mathbb{C}$. $W = C_2$, due to b^2 above.

$\cdot$ $X = Nil(\mathfrak{g}) \rightsquigarrow \mathcal{D}$ = Weyl fan (collection of root hyperplanes), W = Weyl group($\mathfrak{g}$).

Thm (Namikawa): $\cdot$ $\mathcal{D}$ is a union of finitely many hyperplanes (through 0);

$\cdot$ There are finitely many crepant (= symplectic) resolutions (or minimal models) ~~of X~~ $p_i : \tilde{X}_i \twoheadrightarrow X$ of X; $i \in I$

$\cdot$ $H^2(\tilde{X})_{\mathbb{R}} \setminus \mathcal{D}_{\mathbb{R}} = \{ w(\mathrm{Amp}(\tilde{X}_i)) \mid i \in I, w \in W \}$.
Real form ($H^2(\tilde{X}, \mathbb{R})$)

Cor: For $Nil(\mathfrak{g})$, and for any Slodowy slice to $\mathcal{O}_x$ $\cap Nil(\mathfrak{g})$, $\exists!$ crepant resolution.

(4/7)

(subtle) Ex of distinct S.R.'s:

$$\begin{tikzcd} \tilde{X}_1 = T^*Gr(n,m) \arrow[dr, two heads] & & T^*Gr(n,n-m) = \tilde{X}_2 \arrow[dl, two heads] \\ & X = \{A \in \mathfrak{sl}_n \mid A^2=0, \mathrm{rk}(A) \le \min(m,n-m)\} & \end{tikzcd}$$

Of course, $T^*Gr(n,m) \xrightarrow{\sim} T^*Gr(n,n-m)$
$V \longmapsto V^\perp$ (using an inner product),

but this isomorphism induces a nontrivial autom. of X.

Thm (Bellamy – Craw): The crepant resolutions of $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ are exactly those given by varying θ:
$= \mathfrak{M}_{0,\theta}(\tilde{Q}_\Gamma, (1, n\delta))$

Q_Γ = McKay ext. Dynkin quiver,
$\tilde{Q}_\Gamma$ = framed version $\to Q_\Gamma$.

One for isomorphism class per $W = C_2 \times W_\Gamma$ orbit of GIT chambers (complement of locus of θ' where $\mathfrak{M}_{0,\theta'}$ singular).

$\Rightarrow$ # S.R.'s $/\cong$ is $\prod_{i=1}^{\ell} \frac{(n-1)h + d_i}{d_i}$,

h = Coxeter # of W_Γ = Weyl group of $(Q_\Gamma)_{\text{finite}}$
d_i = degrees of fund. invariants, i.e. generators of $\mathbb{C}[\mathfrak{h}_\Gamma]^{W_\Gamma}$.

Use "linearisation map": Send character $\theta: GL_{(1,n\delta)} \to \mathbb{C}^\times$ to associated line bundle class on $\tilde{X}$.

Thm (Bellamy – S.): ALL quiver varieties are symp. sings.
Classify which admit S.R.'s: they are products of those of types: • a point • $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ (from $Q_\Gamma, n\delta$) • O'Grady example and related

indivisible (non-isotropic imaginary) dim vector

Compute cell Weyl groups.

In progress w/ Craw: also classify resols (from GIT usually?).

Use: symplectic Kirwan surjectivity: $H^*_{G_\alpha}(pt) \cong H^*_{G_\alpha}(\mu^{-1}(0)) \longrightarrow H^*_{G_\alpha}(\mu^{-1}(0)^{ss}) \cong H^*(\mathfrak{M}_{0,\theta})$, for degree ≤ 2.

Symplectic Duality: $X \longleftrightarrow X^!$

$\mathcal{D} \subseteq H^2(\tilde{X}) \longleftrightarrow \mathcal{E} \in \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$

Here $\mathbb{T}$ = a max torus acting ~~symplectically~~ Hamiltonianly, commuting with dilation

$\mathcal{E}$ = locus of cocharacters of $\mathbb{T}$ with infinite fixed point sets.

Quantisation:

Thm (Losev) If X is a conical symplectic singularity, then also $H^2(\tilde{X})/W$ parameterises filtered quantisations = filtered noncommutative deformations of $\mathcal{O}(X)$ (recovering S_7–7).

$\Leftrightarrow$ All quantisations arise as $\Gamma(\tilde{X}, \mathcal{D})$, $\tilde{X} \to X$, i.e. by quantising resolution (or partial resolution: minimal model).

By aforementioned Kirwan surjectivity (McGerty + Nevins) for quiver varieties, this should also imply:

- Universal filtered quantisation of $\mathfrak{M}_{0,0}(Q,\alpha)$ is given by

Losev ⇒ (+ Namikawa) $\mathcal{D}(\mathrm{Rep}_\alpha Q) /\!/\!/_\lambda GL_\alpha$.

Namikawa ⇒ · Universal filtered commutative deformation is $\mathfrak{M}_{0,\lambda}(Q,\alpha)$

⇓ · All crepant resolutions are of form $\mathfrak{M}_{\theta,0}(Q,\alpha)$

for suitable α [some technical conditions in Bellamy–S.]

i.e. case where $\mathfrak{M}_{\theta,0}(Q,\alpha)$ is a resolution for generic θ.

(Otherwise can do other things: products, blow-up, etc, to resolve.)

$\mathfrak{g}$ = f.d. semisimple Lie algebra

⇝ Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of reps:
(more interesting than just f.d.):

V s.t. • Borel $\mathfrak{b} \subseteq \mathfrak{g}$ acts locally finitely
• Cartan $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ acts semisimply (= sum of 1-dim reps)

Can also fix $\chi : Z(U\mathfrak{g}) \to \mathbb{C}$ character, and
$\mathcal{O}_\chi := \{V$ as above, $\forall v \in V$, $(\ker \chi)^{N} v = 0$ for $N \gg 0\}$
generalised central character χ. ⇒ at most one f.d. rep!

Ex: $\mathfrak{g} = \mathfrak{sl}_2$: $\mathcal{O}_{\chi_0}$ contains: • trivial rep $\mathbb{C}$ • extensions of these
$[\chi_0(Z(U\mathfrak{g}) \cap \mathfrak{g} \cdot U\mathfrak{g}) = 0]$ • Verma V_{-2}

Verma module $V_\lambda := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \underbrace{\mathbb{C}_\lambda}_{\text{rep of } \mathfrak{b}} := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$.

In general, V_λ, $V_{-2-\lambda}$ in same category.

$\lambda = -1$: "singular" case: V_{-1} is unique simple.
$\Rightarrow \mathcal{O}_{-1} \simeq \mathrm{Vect}$, but
$U\mathfrak{g} / \ker \chi_{-1}$ has infinite global dimension
(not true for other values λ!).

General (BLPW) defns of cat $\mathcal{O}$: $\tilde{X} \to X$, $\mathcal{D}$, $D = \Gamma(\tilde{X}, \mathcal{D})$
$\mathcal{O}_a$ "additive": Reps of D which are locally finite wrt. D_+
dilations act with + weight

$\mathcal{O}_g$: "geometric": $\mathcal{D}$-mods set-theoretically supported on attracting fixed locus of $(\mathbb{C}^\times)$ on $\tilde{X}$, + satisfy a technical cond (admit lattice for subalg $\mathcal{D}(0) \subseteq \mathcal{D}$)

Thm (Beilinson–Ginzburg–Soergel):
Cat $\mathcal{O}_{x_0}$ for $\mathfrak{g}$ is Koszul self dual

Defn $\mathcal{O}$, $\mathcal{O}^!$ are (weakly) Koszul dual if

$$\mathcal{O}^! \simeq \mathrm{Ext}^\bullet(\oplus L_i, L_i)\text{-mod}$$
L_i = simples in $\mathcal{O}$

and vice-versa.

Symplectic Duality "conjecture" (proved under some assumptions in Webster for Higgs + Coulomb branch):
$\mathcal{O}_g(X)$, $\mathcal{O}_g(X^!)$ Koszul dual.

Exs of the Symplectic Duality:

3D mirror symmetry: Higgs branch quiver vars . Coulomb branch

$T^*V/\!/\!/G$ Math: [BFN]

$S_{\mu\nu} \longleftrightarrow S_{\nu^t \mu^t}$ · Hypertoric $T^*V/\!/\!/T$ torus Gale dual $T^*V/\!/\!/T^\vee$ "

SB of type A